ST440/540 Applied Bayesian Analysis
Lab activity for 3/31/2025

Announcements

- Final homework assignment is due this Friday, April 4.
- Abstract is due 4/11.
- I will send the exam later this week. It is due April 14.

A. STUDENT QUESTIONS

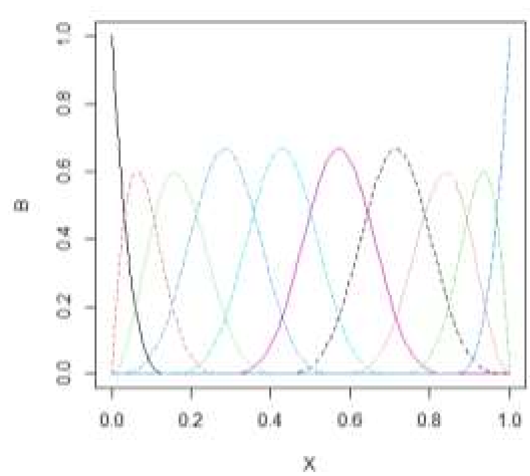
(1) How does R/JAGS view the nonparametric modeling different than the normal method?

JAGS does not use different algorithms for specific models/methods. It just looks at the full conditional for each parameter and decides on an MCMC algorithm to use. This is the great thing about JAGS, you can you it for anything, although it may not the most efficient method for anything.

(2) What does spline mean? I'm just not familiar with the intuition behind a lot of the terms used in this week's videos.

A spline is just a math formula that takes in one value, X, and spits out several values, B(X). These B values can then be used as covariate to allow for the effect of X to be nonlinear.

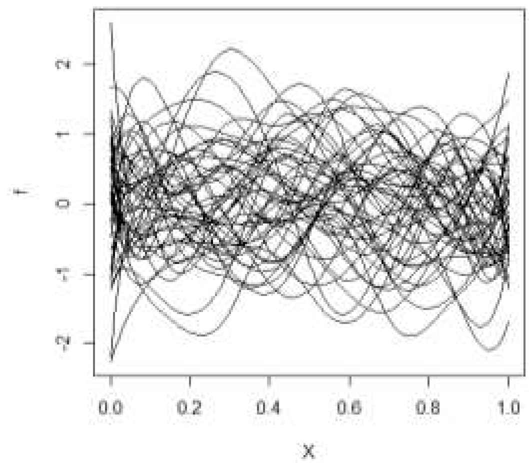
```
library(splines)
X <- seq(0,1,.01)
B <- bs(X,10,intercept=T)
matplot(X,B,type="l")
length(X)
[1] 101
dim(B)
[1] 101  10
```



(3) What does a prior on a function look like, and how would you choose that?

It is hard to visualize the PDF, so the best way is probably to make several draws from the prior. Here are 50 draws from the prior f(X) = \sum_l B_l(x)alpha_l where alpha_l \sim N(0,1). From these draws you can get an idea of what the prior is on say the range of f(X) for different X and the smoothness of the curve.

```
f <- matrix(0,101,50)
for(i in 1:50){f[,i] <- B%*%rnorm(10)}
matplot(X,f,type="l",lty=1,col=1)
```



(4) When using Bayes Factors and SSVS in conjunction with MCMC do you have to be able to write all the models you test as forms or variations of each other? What do you do if they cannot be written this way?

Yes, all models you consider must be nested in one large model. In some cases like linear regression this is natural, but in others is it not and this case I would probably use a different approach like DIC/WAIC/CV.

(5) Would you usually want to use Jeffery's prior when using Bayes Factors due to the sensitivity to the prior? If not, what is your approach to selecting a reasonable prior?

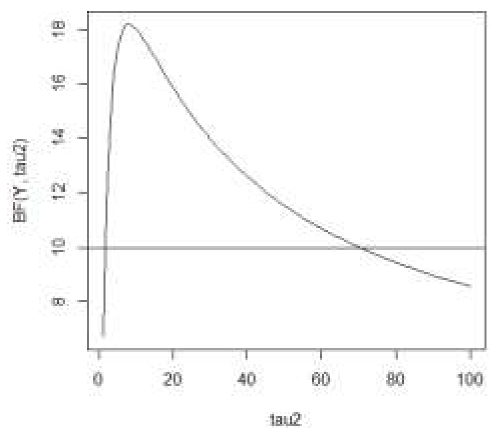
Jefferys' prior is a solid choice, but you have to be careful that the prior is proper otherwise the Bayes factors is not well defined. The approach to selecting a prior is the same as other cases, it requires some knowledge of the problem to set a reasonable range on parameters. Unlike say 95% intervals, Bayes factors are almost always sensitive to the prior so it is important to apply several priors and show how the results are sensitive to this choice.

(6) Can we go an example of a Bayes factor being misleading in the case of one model having a poorly chosen prior?

The classic example is Y|mu ~ N(mu,1) with prior mu ~ Normal(0,tau^2). Then the BF of the null hypothesis that mu=0 versus the alternative that mu \ne 0 is given in the BF function below. If we observe Y=3, then this is 3 SDs above the mean under the null and a frequent p-value would be small so we'd reject the null. On the other hand, if we reject the null if BF>10, then according to the plot below we would reject if tau^2 is between about 5 and 70, which is a head-scratcher.

```
BF <- function(Y,tau2){
  exp(0.5*(Y^2)*tau2/(1+tau2))/sqrt(1+tau2)
}

Y    <- 3
tau2 <- seq(1,100,1)
plot(tau2,BF(Y,tau2),type="l")
abline(10,0)
```



B. HOMEWORK AND QUIZ SOLUTIONS

Assume Y|theta ~ Binomial(n,theta). We wish to test

Model 1: theta=0.5

Model 2: theta is not 0.5 and follows a uniform prior.

We observe n=10 and Y= 5, which gives Bayes factor approximately equal to 5. Describe what this Bayes factor means and the decision you would reach comparing Models 1 and 2.

A Bayes factor of 5 means that the posterior probability of Model 2 is five times larger than Model 1. Usually, the threshold is 10 to be considered strong enough to reject the Model 1 in favor of Model 2.

C. DISCUSSION QUESTIONS

(1) Let Y be the party affiliation of a voter and X be their annual income. Say Y is either R, D or I and X is continuous and positive. The goal is to build a model to predict party affiliation given income.

(a) Below are two modeling approaches based on logistic regression. Which do you prefer and why?

Model 1

$\text{Logit}[\text{Prob}(Y=R)] = a_1 + b_1 * X$ $\text{Logit}[\text{Prob}(Y=D)] = a_2 + b_2 * X$ $\text{Logit}[\text{Prob}(Y=I)] = a_3 + b_3 * X$

Model 2

$\text{Logit}[\text{Prob}(Y=R)] = a_1 + b_1 * X$ $\text{Logit}[\text{Prob}(Y=D | Y \neq R)] = a_2 + b_2 * X$

Model 1 doesn't make sense because the three probabilities don't necessarily sum to one.

(b) In the second model, what is the probability that a voter with income X=x is independent? Use notation $\text{Logit}(\text{Prob}(Y=R)) = a_1+b_1X \iff \text{Prob}(Y=R) = \text{expit}(a_1+b_1X)$.

Prob(Y=I) = Prob(not R)* Prob(not D|not R) = [1-expit(a1+b1X)] [1-expit(a2+b2X)].

(c) In the second model, what is the interpretation of the parameter b_2?

Given that Y is either D or I, the log odds of D increase by b2 if X increases by 1.

(d) How would you modify this model if the covariate was X discrete with levels low, medium and high?

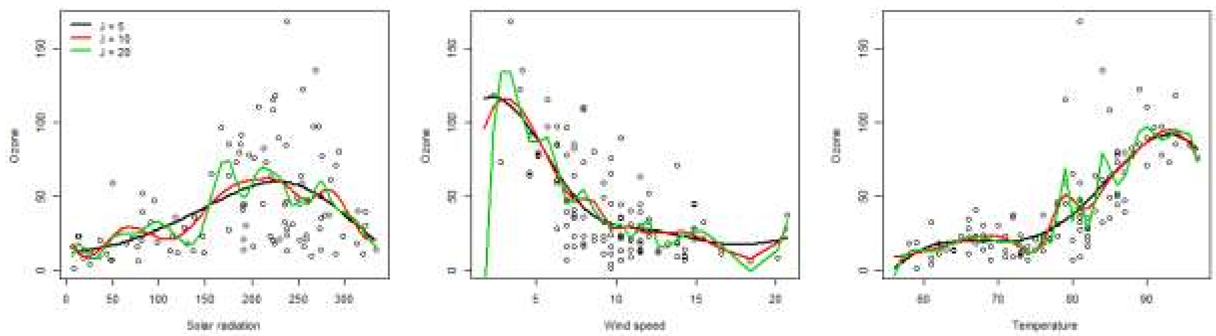
Add two dummy variables, X1 = 1 if medium and X1 = 0 otherwise and X2 = 1 if high and X2 = 0 otherwise,

$\text{Logit}[\text{Prob}(Y=R)] = a1 + b1*X1 + c1*X2$ $\text{Logit}[\text{Prob}(Y=D | Y \neq R)] = a2+b2*X1+ c2*X2$

(2) The plots below show the fit of a non-parametric regression model with

$$Y_i = a + \sum_{j=1}^{J} B_j(X_i) b_j + e_i$$

and flat priors for the regression coefficients $a, b_1, \ldots, b_J$ The three plots use the same response variable Y but different X variables. The code is on the final page



(a) Visually, which values of J look the best for each fit?

I'd say 5 or 10.

(b) How would you formally select J?

Cross validation, DIC or WAIC are all options.

(c) If the flat priors were replaced by normal priors $b_j \sim \text{Normal}(0,v)$ with $v \sim \text{InvGamma}(0.1,0.1)$, would you expect to need more or fewer basis functions? Why?

I'd expect we'd need more basis functions because the prior would prevent over-fitting so we could have more basis function and retain a stable fit.

(3) Consider the models

M1: Y ~ N(0, σ^2)

M2: Y|μ ~ N(μ, σ^2) and μ ~ N(0, cσ^2)

The Bayes factor comparing M2 and M1 is

$$BF = \frac{1}{\sqrt{1+c}} exp\left\{-\frac{y^2}{2\sigma^2}\left(\frac{c}{c+1}\right)\right\}$$

(a) What happens to the Bayes factor as c -> infinity?

The prior becomes more uninformative and BF goes to zero, favoring the null model.

(b) What does this tell you about Bayes factors?

They are very sensitive to the prior.

(4) Take a minute to review the analysis of the Gambia data,

https://www4.stat.ncsu.edu/~bjreich/BSMdata/SSVS.html

Below is output from the SSVS model and Bayesian logistic regression with uninformative Gaussian priors for all parameters

SSVS model

	Inc_Prob	**50%**	**5%**	**95%**
Age	1.00	0.26	0.19	0.34
Netuse	1.00	-0.25	-0.34	-0.17
Treated	0.79	-0.13	-0.24	0.00
Green	1.00	0.29	0.21	0.37
PCH	0.56	-0.05	-0.19	0.00

Flat priors

	Mean	**SD**	**5%**	**95%**
Age	0.27	0.05	0.18	0.37
Netuse	-0.25	0.05	-0.36	-0.15
Treated	-0.13	0.06	-0.25	-0.01
Green	0.29	0.05	0.19	0.39
PCH	-0.10	0.05	-0.20	0.01

(a) How do the results compare? Which model would you use?

The model fits are pretty similar, so I'd probably us flat priors because it's faster and easier to explain.

(b) We have now used two ways to determine if a covariates is "significant": (i) SSVS and inclusion probabilities>0.5 and (ii) a flat prior and seeing if zero is included in the posterior intervals. What are the pros and cons of these two approaches?

For models with only a few covariates I use posterior intervals, but if there are many covariates it's better to use SSVS.

(5) The data generated below has very strong collinearity.

(a) What do you anticipate the output of the SSVS model will be in tables below?

	Inc_Prob	50%	5%	95%
X1	-	-	-	-
X2	-	-	-	-
X3	-	-	-	-

Model	Posterior probs
NULL	-
X1	-
X2	-
X3	-
X1 + X2	-
X1 + X3	-
X2 + X3	-
X1 +X2 +X3	-

The values are

```
Inc_Prob 50% 5% 95%
beta[1] 0.58 0.0 -5.23 1.55
beta[2] 0.65 0.4 -0.49 7.10
beta[3] 0.51 0.0 -0.67 1.51
```

Model probabilities:

```
Intercept + X1 + X2 Intercept + X2 Intercept + X1 + X2 + X3
0.201 0.173 0.144
Intercept + X2 + X3 Intercept + X3 Intercept + X1
0.132 0.118 0.117
Intercept + X1 + X3
0.114
```

(b) In real life, how would you handle this analysis?

Remove some covariates until the model has less collinearity.

Code

```
n  <- 100
p  <- 3
set.seed(919)
X1 <- rnorm(n)
X2 <- X1 + 0.01*rnorm(n)
X3 <- X2 + 0.01*rnorm(n)
X  <- cbind(X1,X2,X3)
Y  <- rnorm(n,X2,1)

> round(cor(X),4)
       X1     X2     X3
X1 1.0000 0.9999 0.9999
X2 0.9999 1.0000 0.9999
X3 0.9999 0.9999 1.0000

> summary(lm(Y~X))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.06191    0.09623  -0.643    0.522
XX1         -8.13549   10.22983  -0.795    0.428
XX2         15.43476   13.65390   1.130    0.261
XX3         -6.13203    9.51324  -0.645    0.521

Residual standard error: 0.9493 on 96 degrees of freedom
Multiple R-squared:  0.5753,    Adjusted R-squared:  0.5621
F-statistic: 43.36 on 3 and 96 DF,  p-value: < 2.2e-16
```

```
# SSVS model in JAGS
m <- textConnection("model{
    for(i in 1:n){
       Y[i]  ~ dnorm(mu[i],taue)
      mu[i] <- alpha + X[i,1]*beta[1] + X[i,2]*beta[2] + X[i,3]*beta[3]
    }
    for(j in 1:3){
        beta[j] <- gamma[j]*delta[j]
        gamma[j] ~ dbern(0.5)
        delta[j] ~ dnorm(0,taub)
    }
    alpha ~ dnorm(0,0.01)
    taub   ~ dgamma(0.1,0.1)
    taue   ~ dgamma(0.1,0.1)
}")

# Run JAGS
   library(rjags)
   data   <- list(Y=Y,X=X,n=n)
   burn   <- 10000
   iters  <- 50000
   chains <- 3
   model  <- jags.model(m,data = data, n.chains=chains,quiet=TRUE)
   update(model, burn, progress.bar="none")
   samps  <- coda.samples(model, variable.names=c("beta"),
                          thin=5, n.iter=iters, progress.bar="none")
   plot(samps)

# Summarize the posterior of beta
   beta    <- NULL
   for(l in 1:chains){
     beta <- rbind(beta,samps[[l]])
   }
   Inc_Prob <- apply(beta!=0,2,mean)
   Q        <- t(apply(beta,2,quantile,c(0.5,0.05,0.95)))
   out      <- cbind(Inc_Prob,Q)
   round(out,2)

# Compute model probabilities
   model <- "Intercept"
   names <- paste0("X",1:3)
   for(j in 1:3){
     model <- paste(model,ifelse(beta[,j]==0,"","+"))
     model <- paste(model,ifelse(beta[,j]==0,"",names[j]))
   }
   model_probs <- table(model)/length(model)
   model_probs <- sort(model_probs,dec=T)
   round(model_probs,3)

# Plot predicted versus fitted
   plot(X%*%colMeans(beta),Y)
```

Code for problem 2

```
library(splines)

data(airquality)
Ozone <- airquality[,1]
SR    <- airquality[,2]
Wind  <- airquality[,3]
Temp  <- airquality[,4]

par(mfrow=c(1,3))
for(i in 1:3){
 if(i==1){X <- SR;xlab <- "Solar radiation"}
 if(i==2){X <- Wind;xlab <- "Wind speed"}
 if(i==3){X <- Temp;xlab <- "Temperature"}

 Y    <- Ozone
 ylab <- "Ozone"

 ooo  <- order(X)
 Y    <- Y[ooo]
 X    <- X[ooo]
 plot(X,Y,xlab=xlab,ylab=ylab)
 m    <- c(5,10,20)
 for(j in 1:length(m)){
   B   <- bs(X,df=m[j])
   b   <- lm(Y ~ B)$coef
   lines(X,b[1] + B%*%b[-1],lwd=2,col=j)
  }
  if(i==1){
    legend("topleft",paste("J =",m),lwd=2,col=1:4,bty="n")
  }
}
```